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Technical Notes

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New Look at Kirchhoff's Theory of Plates

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Nomenclature

a, b	=	plate lateral dimension
D	=	flexural stiffness of the plate/unit width
E	=	Young's modulus
E'	=	$E/(1-\nu^2)$
G	=	modulus of rigidity
$q(x, y)$	=	applied load density
T_x, T_{xy}, T_{xz}	=	prescribed stress distributions along the $x =$ constant edges
u, v	=	in-plane displacements in the x and y directions
u_1, v_1, w_0	=	midplane displacements in x, y, and z directions, respectively
w	=	vertical deflection
ε_z, γ_{xz}, γ_{yz}	=	transverse strains
ν	=	Poison's ratio
σ_x, σ_y, τ_{xy}	=	bending stresses
τ_{xz}, τ_{yz}	=	transverse shear stresses
τ^*_{xz}, τ^*_{yz}, σ^*_z	=	statically equivalent stresses
φ_1	=	harmonic function that represents in-plane
$2h$	=	plate thickness
∇^2	=	Laplace operator

Introduction

KIRCHHOFF'S theory [1] and the first-order shear deformation theory (FSDT) [2] of plates in bending are simple theories and continuously used to obtain design information. Within the classical small deformation theory of elasticity, the problem consists of determining three displacements, u, v, and w, that satisfy three equilibrium equations in the interior of the plate and three specified surface conditions. FSDT is a sixth-order theory with a provision to satisfy three edge conditions and maintains, unlike in Kirchhoff's theory, independent linear thicknesswise distribution of tangential displacement even if the lateral deflection, w, is zero along a supported edge. However, each of the in-plane distributions of the transverse shear stresses that are of a lower order is expressed as a sum of higher-order displacement terms. Kirchhoff's assumption of zero transverse shear strains is, however, not a limitation of the theory as a first approximation to the exact 3-D solution. The main limitation in Kirchhoff's theory is due to in-plane displacements expressed as gradients of lateral displacement, resulting in a fourth-order theory with two edge conditions instead of a sixth-order theory with three edge conditions, as demanded by Poisson. The Poisson–Kirchhoff boundary-condition paradox is well known and has not been properly resolved over the past 15 decades or so, even though several sixth-order shear deformation theories are reported in the literature. Here, a proper resolution of this paradox is presented without considering either the shear energy due to transverse shear deformations or higher-order approximations to displacements.

Primary Flexure Problem

For simplicity in presentation, a rectilinear domain $0 < x < a$, $0 < y < b$, and $-h < z < h$ with reference to Cartesian coordinate system (x, y, z) is considered (see Fig. 1). The thickness, $2h$, of the plate is small compared with its lateral dimensions, a and b. The material of the plate is homogeneous and isotropic, with elastic constants E (Young's modulus), ν (Poison's ratio), and G (modulus of rigidity) that are related to each other by $E = 2(1+\nu)G$.

The plate is subjected to asymmetric load $q(x, y)$ and zero shear stresses along the top and bottom faces, that is, along $z = \pm h$ faces:

$$\sigma_z = \pm q/2 \qquad \tau_{xz} = 0 \qquad \tau_{yz} = 0 \tag{1}$$

We assume that the edge conditions are prescribed such that the in-plane displacements (u, v) and bending stresses $(\sigma_x, \sigma_y, \tau_{xy})$ are antisymmetric in z, and the vertical deflection w and transverse shear stresses (τ_{xz}, τ_{yz}) are symmetric in z. We consider here a primary flexure problem in which the edge conditions are specified in the following form:

$$u = 0 \quad \text{or} \quad \sigma_x = zT_x(y) \tag{2a}$$

$$v = 0 \quad \text{or} \quad \tau_{xy} = zT_{xy}(y) \tag{2b}$$

$$w = 0 \quad \text{or} \quad \tau_{xz} = \tfrac{1}{2}(h^2 - z^2)T_{xz}(y) \tag{2c}$$

along an $x =$ constant edge and analogous conditions along a $y =$ constant edge. In these edge conditions, T_x, T_{xy}, and T_{xz}, are prescribed distributions along the $x =$ constant edges and analogous distributions along the $y =$ constant edges. In addition to these conditions (1) and (2), stress components in the interior of the plate have to satisfy equilibrium equations:

$$\sigma_{x,x} + \tau_{xy,y} + \tau_{xz,z} = 0 \tag{3a}$$

$$\sigma_{y,y} + \tau_{xy,x} + \tau_{yz,z} = 0 \tag{3b}$$

$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} = 0 \tag{3c}$$

in which the subscript after the comma denotes a partial derivative operator.

In displacement-based models, the six stress components are expressed in terms of displacements via six stress-strain constitutive relations and six strain-displacement relations. We treat the aforementioned problem within the classical small deformation theory of elasticity.

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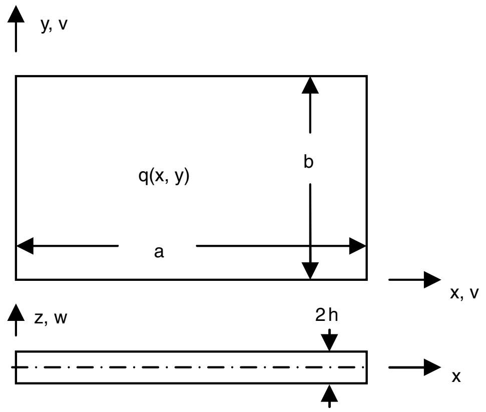

Fig. 1 Plate configuration and coordinate system.

Analysis

In Kirchhoff's theory, transverse strains ε_z, γ_{xz}, and γ_{yz} are assumed to be zero and σ_z is neglected in the constitutive relations. From strain-displacement relations, the former assumption implies that lateral deflection w is independent of z denoted by w_0 and in-plane displacement u and v are linear in z, that is,

$$u = zu_1(x, y) \qquad v = zv_1(x, y) \tag{4}$$

so that

$$u_1 + w_{0,x} = 0 \qquad v_1 + w_{0,y} = 0 \tag{5}$$

From the latter assumption, in-plane stress-strain relations are

$$\varepsilon_x = (1/E)(\sigma_x - \nu\sigma_y) \qquad \varepsilon_y = (1/E)(\sigma_y - \nu\sigma_x) \qquad \tau_{xy} = G\gamma_{xy} \tag{6}$$

in which $\varepsilon_x = u_{,x}$, $\varepsilon_y = v_{,y}$, and $\gamma_{xy} = (v_{,x} + u_{,y})$. Transverse stress-strain relations are ignored due to the assumption of zero transverse strains. From the first two relations in Eq. (6), bending stresses σ_x and σ_y in terms of ε_x and ε_y are

$$\sigma_x = E'(\varepsilon_x + \nu\varepsilon_y) \qquad \sigma_y = E'(\varepsilon_y + \nu\varepsilon_x) \tag{7}$$

in which $E' = E/(1 - \nu^2)$.

In the present analysis, u_1 and v_1 are considered as unknown functions instead of w_0 and express Eq. (5) in the form

$$w_0 = -\int (u_1 \mathrm{d}x + v_1 \mathrm{d}y) \tag{8}$$

and

$$v_{1,x} - u_{1,y} = 0 \tag{9}$$

ensuring continuity of the in-plane variation of w_0. The transverse stresses neglected earlier are obtained from the thicknesswise integration of Eq. (3). By using Eqs. (4), (6), (7), and (9) and satisfying zero transverse shear stresses in Eq. (1), these stresses, known as statically equivalent stresses and denoted by τ^*_{xz}, τ^*_{yz}, and σ^*_z, are obtained as

$$\tau^*_{xz} = \tfrac{1}{2}(h^2 - z^2)E' e_{1,x} \tag{10a}$$

$$\tau^*_{yz} = \tfrac{1}{2}(h^2 - z^2)E' e_{1,y} \tag{10b}$$

$$\sigma^*_z = -\tfrac{1}{2}(h^2 z - z^3/3)E' \nabla^2 e_1 \tag{10c}$$

in which $e_1 = (u_{1,x} + v_{1,y})$, and ∇^2 is the Laplace operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$.

The vertical load condition in Eq. (1) gives, from Eq. (10c),

$$\nabla^2 e_1 + q/D = 0 \tag{11}$$

in which $D = (2/3)Eh^3/(1 - \nu^2)$. Equations (9) and (11) are the governing equations for determination of u_1 and v_1.

For comparison with Kirchhoff's theory, it is convenient to assume u_1 and v_1 in the form

$$u_1 = -(w_{0,x} - \varphi_{1,y}) \qquad v_1 = -(w_{0,y} + \varphi_{1,x}) \tag{12}$$

Equations (9) and (11) in terms of φ_1 and w_0 are

$$\nabla^2\varphi_1 = 0 \qquad \nabla^2\nabla^2 w_0 = q/D \tag{13}$$

which is a sixth-order system of equations. It can be seen that the present analysis differs from Kirchhoff's theory in introducing harmonic function φ_1 in the representation of in-plane displacements. This function, φ_1, and w_0 are coupled only through edge conditions (2a) and (2b). If w_0 is zero all along the wall of the plate, φ_1 is identically zero and the analysis coincides with Kirchhoff's theory.

It is to be noted that the present analysis, unlike in Kirchhoff's theory, has the provision to satisfy prescribed τ_{xy} along the edge of the plate, even in the case of vertical load q and when the applied transverse shear stresses are zero (in St. Venant's torsion problem, in-plane shear is associated with transverse shear and warping function u or v is harmonic. Here, φ is harmonic but none of the displacements are harmonic). The present estimates to displacements maintain the same order of accuracy as in Kirchhoff's theory.

It is quite clear that Kirchhoff's approximations of in-plane displacements modified with the addition of gradients of a harmonic function are adequate to derive a sixth-order system of equations. With reference to the 3-D solution, the limitations of the analysis, as in Kirchhoff's theory, regard the transverse shear stress-strain relations and the neglect of σ_z in constitutive relations. However, we note that

$$\tau_{xz} = G\varphi_{1,y} \qquad \tau_{yz} = -G\varphi_{1,x} \tag{14}$$

are reactive stresses due to prescribed τ_{xy} along an edge of the plate. They are self-equilibrating reactions in the interior due to zero shear conditions in Eq. (1) and do not effect τ^*_{xz} and τ^*_{yz} in Eqs. (10a) and (10b). From Eqs. (3c) and (14), one obtains the linear variation of σ_z in z is zero, eliminating Kirchhoff's assumption of neglecting σ_z in constitutive relations. Furthermore, the contracted edge condition is avoided due to the decoupling of in-plane and vertical shears along the edges without recourse to shear deformation theories.

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